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REPLY

Reply to Comment on 'Excited states in the infinite quantum lens potential: conformal mapping and moment quantization methods'

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Abstract
The suitability of conformal transformation (CT) analysis, and the eigenvalue moment method (EMM), for determining the eigenenergies and eigenfunctions of a quantum particle confined within a lens geometry, is reviewed and compared to the recent results by Even and Loualiche (2003 *J. Phys.: Condens. Matter* **15** 8465). It is shown that CT and EMM define two accurate and versatile analytical/computational methods relevant to lens shaped regions of varying geometrical aspect ratios.

Response

The recent results of Even and Loualiche (EL) (2003) consider a specialized lens geometry which allows one to generate separable solutions, but only for a fixed height/width ratio of $b/a = 1/2$. Although their results are of theoretical/pedagogical interest, the needs of the research community exceed these restrictions, as cited in the work by Rodríguez, Handy and Trallero-Giner (RHT) (2003); therefore, RHT emphasize those formalisms that offer arbitrary flexibility with respect to the same geometrical parameter: $0 < b/a \leqslant 1$.

Consistent with this, an additional objective of the RHT analysis was to develop effective methods for studying the highly singular, and theoretically important, *thin lens* limit, $b/a \to 0$, for which conventional approaches may be inadequate. This thin lens limit was partially studied by RHT and discussed in greater detail with respect to the ground state energy levels, within each azimuthal symmetry class, in the related work by Handy *et al* (2001). Clearly, none of this is possible within the EL formalism.

Two approaches are considered by RHT. The first involves conformal transformation (CT) analysis, which defines a very powerful framework for tackling the various geometrical ratios alluded to. The second is the eigenvalue moment method (EMM), which is a, relatively novel, computational procedure, based on important mathematical theorems (i.e. originating from the classic *moment problem*), for obtaining accurate discrete state energy values through the generation of converging (lower and upper) bounds. It is well known that many eigenenergy estimation methods can lead to significantly varying results when they are used to analyse highly singular systems. The ability to generate tight, lower and upper eigenenergy bounds, provides an unequivocal procedure by which to accurately discriminate between competing theories.

The CT approach has been used on other important problems, as cited in the work by RHT (particularly the works by Robnik 1984, Berry 1986, Berry and Robnik 1986a, 1986b and Itzykson *et al* 1986). Other relevant, and more recent, references which apply these methods to lens geometries of varying proportions include the works by Muñoz *et al* (2003), Rodríguez and Trallero-Giner (2002) and Trallero-Herrero *et al* (2001). In particular, the work by Rodríguez and Trallero-Giner (2002) also considers the extension of CT methods to lens shaped systems perturbed by an external potential. In this case, one must work with Dirichlet boundary conditions. The CT analysis generates the required unperturbed states, leading to a perturbative expansion for the new system.

With regard to the lens geometry considered by RHT, the CT analysis allows one to generate important physical parameters of the system, such as transition probabilities, optical strengths and other optical properties, energy variations with respect to external fields, etc. All of these quantities can be conveniently expressed through such expansions (involving Bessel functions). With regards to the computational aspects of this approach (in response to the implied critique by EL), the diagonalization of symmetric matrices of dimensions $<150 \times 150$, is, nowadays, not a complicated process. Accordingly, the CT approach is quite appropriate, versatile, expedient and accurate in determining the discrete state wavefunctions and energies for these important systems of *varying* geometry. Thus, we disagree with the interpretation offered by EL with regards to the utility of CT methods.

The second approach discussed by RHT, the EMM formalism, has been used on various singular perturbation type problems. The most significant of these is the quadratic Zeeman effect for superstrong magnetic fields (Handy *et al* 1988), often referred to as the 'last unsolved problem in atomic physics'. The EMM analysis generated impressive bounds for the binding energy of the ground state (which is the most singular state). It defined an elegant and simple computational algorithm (coincidentally in parabolic coordinates) for calculating tight bounds for arbitrary magnetic field strengths. Interestingly, these results corroborated a more complicated analysis by Le Guillou and Zinn-Justin (1983) based on an order-dependent conformal transformation analysis.

Of further importance is the fact that a moment representation based analysis (i.e. EMM) has been shown to be directly related to scaling transform theory, which in turn underlies wavelet analysis (Handy and Murenzi 1998, 1999). Indeed, the EMM procedure can be regarded as an affine map invariant variational procedure. This means that the EMM eigenenergy results automatically take into account all possible affine map transformations (i.e. scalings, translations, rotations, inversions, etc) within the variational sample function space. This makes EMM analysis particularly sensitive to multiscale features of singular systems and thus explains both its numerical robustness and accuracy.

In our opinion, it is premature to pass judgement, at this stage, on which analytical/computational approaches will ultimately suit the practical needs of the researchers involved. Certainly, the lack of geometrical flexibility within the particular representation defined by EL's analysis limits its relevance, in this regard, in contrast to the versatility of the CT and EMM approaches, as presented by RHT.

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